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Technical Note

Solution of the Graetz–Brinkman problem with the Laplace transform Galerkin method

Peter P. Valkó

Department of Petroleum Engineering, A&M University, 501K Richardson Building, 3116 Tamu, College Station, TX 77843 3116, USA

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Abstract

The present study concentrates on the effects of viscous dissipation in laminar forced convection. A power law fluid rheology model is applied and the effect of heat conduction in the axial direction is considered negligible. The physical properties are considered constant. Assuming fully developed velocity profile, the development of the temperature profile and its asymptotic behavior are investigated. For the solution of the problem the Laplace transform Galerkin technique is used. The method allows for the most general boundary conditions. A detailed comparison with previously published results provides a verification of the numerical technique. An important feature of the approach is that derivatives and integrals with respect to the axial location can be obtained through the operational rules of the Laplace transformation and hence no numerical derivation or integration is needed. As an application of the numerical model, we focus on the natural cooling regime, when the viscous dissipation of energy is counter-balanced by keeping the wall temperature at the ambient value. We derive a correlation for the asymptotic behavior of the Nusselt number in the natural cooling regime. This correlation reproduces the known value for the Newtonian case and provides a convenient means to normalize the Nusselt number for a wide range of flow behavior indices.

Keywords: Graetz problem; Brinkman problem; Power law rheology; Forced convection; Viscous dissipation; Natural cooling; Laplace transform inversion; Galerkin method

1. Introduction

For the processing of polymer solutions and melts the following heat transfer problem is of particular interest. Fluid at ambient temperature with a well developed laminar velocity profile enters a circular pipe whose wall may be maintained at constant temperature, or cooled (heated) with a constant flux. Heat conduction in the axial direction is negligible in comparison with the heat transport by the over-all fluid motion. Viscous heating is not negligible and the rheology of the fluid is described by a power law. The physical properties can be considered constant. We are concerned with the development of the temperature profile and its asymptotic behavior.

Considering only Newtonian behavior and neglecting the effect of viscous dissipation, this is the well known Graetz–Nusselt problem. It has been thoroughly investigated for the case when the boundary condition is in Dirichlet form (constant wall temperature) and when it is in the Neumann form (constant heat flux). Results are summarized for instance in [1] and [2]. Occasionally

E-mail address: p-valko@tamu.edu

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Nomenclature

Br	Brinkman number
c_1, c_2, c_3	constants in boundary condition
c_p	specific heat (J/kg K)
Gz	Graetz number
h	heat transfer coefficient (W/m^2 K)
M	number of terms in Gaver–Wynn–Rho method
n	number of terms in Galerkin method
Nu	Nusselt number
$p(r)$	polynomial in Galerkin method
Pe	Peclet number
q_w	wall heat flux (W/m^2)
r	dimensionless radial location
R	radial location(m)
R_w	pipe radius (m)
s	Laplace variable
t	dimensionless temperature
T	temperature (K)
T_0	ambient (entrance) temperature (K)
T_w	wall temperature (K)
u	dimensionless velocity
U	axial velocity (m/s)
U_m	mean (superficial) velocity (m/s)
x	dimensionless axial location
X	axial location (m)

Greek symbols

κ	consistency parameter (Pa s$^\nu$)
ν	flow-behavior index
λ	thermal conductivity (W/m K)
ρ	density (kg/m^3)
τ_{RX}	shear stress (Pa)

Subscripts

a	at adiabatic regime
b	bulk
H	at constant wall flux
m	mean
T	at constant wall temperature
x	local
w	wall
1	reference
0	ambient (also: at natural cooling regime)
∞	asymptotic

the so called third boundary condition (Robin form) is also included, [3].

Brinkman [4] brought attention to the importance of viscous dissipation and Lyche and Bird [5] were the first to consider fluids with power law behavior. Parallely, the Graetz series solution was perfected by Brown [6]. This was followed by applying various numerical methods to the non-Newtonian problem, [7–10] and asymptotic expansion techniques [11,12]. Recent and highly reliable results for the power law case (without viscous dissipation) are available in Johnston [13]. Asymptotic behavior for power law behavior with viscous dissipation was considered by Barletta [14]. Other than Newtonian and power law behavior has been also studied, for instance Bingham plastic behavior in [15–17] and recently Phan–Thien–Tanner (PTT) rheology, [18]. The effect of slip at the wall was included, for example, in [19,20]. Another direction of research has been to incorporate axial heat conduction, basically extending the scope of the original analytical approach of Graetz, [21–25] and to determine when the axial conduction can be neglected [13].

This work departs from previous studies in the following. It uses the the Galerkin (weighted residual) method in combination of the Laplace transform. We are aware of the application of the Galerkin method to the Graetz problem [26], but not in Laplace space. Our method allows for a more general form of the boundary condition including the special cases when the constant flux is zero (adiabatic) and when the constant wall temperature coincides with the ambient temperature (natural cooling). In addition, the so called third type of boundary condition is also a special case of our general boundary condition. One important advantage of the approach is that all differentiation and integration are handled analytically in the radial direction and through the operational rules of Laplace transform in the axial direction. High accuracy of the final results is made possible by a novel numerical Laplace transform inversion technique.

2. Governing equations

The problem under consideration is to find the temperature T as a function of axial location X and radial position R. The fluid has a fully developed laminar velocity profile, $U(R)$ corresponding to the power law rheology:

$$\tau = \kappa\left(\frac{\partial U}{\partial R}\right)^{\nu} \tag{1}$$

The energy equation includes the heat generated by the internal friction of the fluid:

$$\rho c_p U(R)\frac{\partial T}{\partial X} = \lambda \frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial T}{\partial R} - \tau_{RX}\frac{\partial U(R)}{\partial R} \tag{2}$$

In (2) the velocity profile is

$$U(R) = U_{\mathrm{m}}\left(\frac{3\nu+1}{\nu+1}\right)\left[1-\left(\frac{R}{R_{\mathrm{w}}}\right)^{\frac{\nu+1}{\nu}}\right] \tag{3}$$

resulting in the derivative:

$$\frac{\mathrm{d}U(R)}{\mathrm{d}R} = U_{\mathrm{m}}\frac{3\nu+1}{\nu}R_{\mathrm{w}}^{\frac{1-\nu}{\nu}}R^{\frac{1}{\nu}} \tag{4}$$

and shear stress:

$$\tau_{\mathrm{RX}} = \kappa R_{\mathrm{w}}^{-\nu-1}\left(\frac{3\nu+1}{\nu}\right)^{\nu}R \tag{5}$$

The primary quantity is the bulk (also called cup-mixing or caloric mean) temperature, defined by

$$T_{\mathrm{b}} = \frac{\int_0^{R_{\mathrm{w}}} RU(R)T\,\mathrm{d}R}{\int_0^{R_{\mathrm{w}}} RU(R)\,\mathrm{d}R} \tag{6}$$

Also of interest is the wall temperature:

$$T_{\mathrm{w}} = T(X, R_{\mathrm{w}}) \tag{7}$$

because it goes into the definition of the local Nusselt number:

$$Nu_{\mathrm{x}} = \frac{2R_{\mathrm{w}}h}{\lambda} = \frac{2R_{\mathrm{w}}\frac{\partial T}{\partial R}|_{R=R_{\mathrm{w}}}}{(T_{\mathrm{w}}-T_{\mathrm{b}})} \tag{8}$$

In order to introduce dimensionless variables, we select a *reference* temperature difference ΔT_1 and define

$$t = \frac{T-T_0}{\Delta T_1} \tag{9}$$

with respect to the ambient temperature, T_0.

The other dimensionless variables are

$$r = \frac{R}{R_{\mathrm{w}}} \tag{10}$$

$$x = \frac{X}{R_{\mathrm{w}}Pe} \tag{11}$$

where

$$Pe = \frac{2R_{\mathrm{w}}U_{\mathrm{m}}\rho c_p}{\lambda} \tag{12}$$

We note that in the literature $2x$ is also used as dimensionless axial coordinate. In some publications the Graetz number is preferred, $Gz = \pi/2x$.

Introducing the dimensionless velocity

$$u(r) = \left(\frac{3\nu+1}{\nu+1}\right)\left[1-r^{\frac{\nu+1}{\nu}}\right] \tag{13}$$

and the Brinkman number

$$Br = \kappa\left(\frac{3\nu+1}{\nu}\right)^{\nu}\frac{U_{\mathrm{m}}^{1+\nu}R_{\mathrm{w}}^{1-\nu}}{\lambda\Delta T_1} \tag{14}$$

we arrive at the dimensionless form of the energy equation:

$$\frac{1}{2}u(r)\frac{\partial t}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial t}{\partial r} - Br\left(\frac{3\nu+1}{\nu}\right)r^{\frac{\nu+1}{\nu}} \tag{15}$$

The boundary conditions will be common for all cases investigated. The entrance temperature is the ambient temperature, therefore

$$t(0,r) = 0 \tag{16}$$

Because of the radial symmetry,

$$\frac{\partial t}{\partial r}(x,0) = 0 \tag{17}$$

The boundary condition at the wall is formulated in the general manner

$$c_1 t(x,1) + c_2\frac{\partial t}{\partial r} + c_3 = 0 \tag{18}$$

where c_1 and c_2 cannot be zero simultaneously. This includes the Dirichlet, the Neumann, and the Robin boundary conditions. Since we allow c_3 to be zero, it also includes the adiabatic regime and the natural cooling regime.

The dimensionless bulk temperature is obtained from

$$t_{\mathrm{b}} = 2\left(\frac{3\nu+1}{\nu+1}\right)\int_0^1 r\left(1-r^{\frac{\nu+1}{\nu}}\right)t\,\mathrm{d}r \tag{19}$$

and will provide most of the required heat transfer quantities directly.

3. Laplace transform Galerkin method

Our approach starts with eliminating the x variable via Laplace transform.

3.1. Equation in Laplace domain

The Laplace transform of the dimensionless temperature (with respect to the variable x) is denoted by $t_s(r)$ with s as the Laplace variable. The transformed energy equation takes the form

$$\frac{1}{2}\left(\frac{3\nu+1}{\nu+1}\right)\left(1-r^{\frac{\nu+1}{\nu}}\right)st_s(r) = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial t_s(r)}{\partial r} - \frac{Br}{s}\left(\frac{3\nu+1}{2\nu}\right)r^{\frac{\nu+1}{\nu}} \tag{20}$$

The entrance condition is absorbed by the transformation, the other boundary conditions become

$$\left.\frac{\partial t_s(r)}{\partial r}\right|_{r=0} = 0 \tag{21}$$

and

$$c_1 t_s(r)|_{r=1} + c_2\left.\frac{\partial t_s(r)}{\partial r}\right|_{r=1} + \frac{c_3}{s} = 0 \tag{22}$$

3.2. Galerkin method

In order to approximate the solution in Laplace space, we introduce the nth order *trial* polynomial $p(r)$, satisfying the two boundary conditions (21) and (22).

$$p(r) = a_0 + \sum_{i=2}^{n-1} a_i r^i - r^n \frac{1}{c_1 + c_2 n} \times \left[c_1 \left(a_0 + \sum_{i=2}^{n-1} a_i \right) + c_2 \sum_{i=2}^{n-1} a_i i + c_3 \right] \tag{23}$$

The $(n-1)$ unknown coefficients are determined from substituting the trial polynomial into (20), then forming the residual ψ:

$$\psi = -\frac{\partial^2 p(r)}{\partial r^2} - \frac{1}{r}\frac{\partial p(r)}{\partial r} + \left(\frac{3v+1}{v+1}\right)\left(1 - r^{\frac{v+1}{v}}\right) s p(r) - \frac{Br}{s}\left(\frac{3v+1}{2v}\right) r^{\frac{v+1}{v}} \tag{24}$$

Taking the partial derivatives of ψ according to the unknown coefficients

$$\phi_i = \frac{\partial \psi}{\partial a_i}, \quad i = 0, 2, 3, \ldots, n-1 \tag{25}$$

and requiring the following integrals to be zero:

$$\int_0^1 \phi_i \psi \, \mathrm{d}r = 0, \quad i = 0, 2, 3, \ldots, n-1 \tag{26}$$

we obtain the system (26) consisting of $(n-1)$ linear equations, from which the $(n-1)$ unknown coefficients (a_i, $i = 0, 2, 3, \ldots, n-1$) can be determined.

For low or moderate n the system can be solved symbolically, obtaining the polynomial coefficients a_i as a function of the Laplace variable s. For $n > 10$ the symbolic solution becomes increasingly cumbersome and hence the linear system (26) is solved numerically, for any s where the coefficients are required.

For differentiation and integration in the r variable we make use of the fact, that the Laplace transform of the temperature is in a form of polynomial in r. Therefore, the Laplace transform of the bulk temperature is easily calculated. The "real domain" value is obtained by numerical inversion of the Laplace transform, that is performing the back-transformation $s \to x$. Similarly, the temperature can be obtained at any location, including the value at the wall.

For differentiation and integration with respect to the x variable we use the operational rules of Laplace transformation. Hence no numerical differentiation/integration is involved in any of our results.

The Galerkin method has been applied in conjunction with the Laplace transform for instance by Sudicky [27] and Sutradhar et al. [28], and a similar combination of Laplace transform with finite elements [31] and boundary elements [29,30] is often applied, though almost exclusively for time dependent problems. In all studies, the obtained Laplace transform has to be inverted numerically, and this step is considered the weakest link in the procedure.

3.3. Numerical Laplace transform inversion

Indeed, inherent in almost all previously suggested numerical inversion algorithms are a number of free parameters, whose selection affects the final results. For early reviews, see Davies and Martin [32] and Narayanan and Beskos [33]. For recent developments we refer to [34–37]. The Gaver–Wynn–Rho algorithm [38] applied in this work overcomes this problem by having only one free parameter: M, the number of terms to be considered. With increasing M, the result converges to the true value. This is made possible by the systematic application of multi-precision computing.

The Gaver–Wynn–Rho algorithm is publicly available in *Mathematica*, [39]. In all our calculations we used the default number of terms $M = 32$. This requires $2.1M$ significant digits in the Laplace transform. The required number of digits could be generously provided by using $4M$ precision during the numerical solution of the linear system.

4. Results and comparison with previous work

First we compare our calculations with the extensive literature available for the case when viscous dissipation can be neglected.

4.1. Constant wall temperature (different from ambient)

Dirichlet boundary condition

$$\Delta T_1 = T_\mathrm{w} - T_0 \tag{27}$$

$$c_1 = 1; \quad c_2 = 0; \quad c_3 = -1 \tag{28}$$

In this case $t_\mathrm{w} = 1$ and if in addition viscous energy dissipation can be neglected, then the change in the bulk temperature is related to the wall heat flux, and hence the local Nusselt number is

$$Nu_{T,x} = \frac{\frac{1}{2}\frac{\mathrm{d}t_\mathrm{b}}{\mathrm{d}x}}{1 - t_\mathrm{b}} \tag{29}$$

where the derivative is obtained directly by inverting $st_{s,\mathrm{b}}$. The mean Nusselt number is the integral average of $Nu_{T,x}$ over the section length, and can be obtained from analytical integration of (29) as

$$Nu_{T,m} = \frac{1}{2x} \ln \frac{1}{1 - t_\mathrm{b}} \tag{30}$$

Table 1
Constant wall temperature, $Br = 0$, $\nu = 1$

x	t_b [6]	t_b [9]	t_b [11]	t_b [13]	t_b Present	$Nu_{T,x}$ Present	$Nu_{T,m}$ Present
0.001	0.038715	0.038247	0.038251	0.03825	0.038250	12.82418	19.50052
0.002	0.059736	0.059659	0.059683	0.05968	0.059682	10.13019	15.38419
0.005	0.106572	0.106451	0.106580	0.10657	0.106572	7.47038	11.26895
0.010	0.163781	0.163482	0.163814	0.16378	0.163781	6.00151	8.94324
0.020	0.248894	0.248405	0.249035	0.24889	0.248894	4.91606	7.15522
0.005	0.421213	0.421350	0.422248	0.42121	0.421213	4.00463	5.46820
0.100	0.604701	0.605891	0.610085	0.60470	0.604701	3.70999	4.64057
0.200	0.810290	0.810310	0.840523	0.81029	0.810290	3.65807	4.15565
0.500	0.978856	0.978675	1.098401	0.97886	0.978856	3.65679	3.85640
1.000	0.999454	0.999514	0.942796	0.99945	0.999454	3.65679	3.75660

In other words, once t_b is calculated all relevant quantities can be obtained. Table 1 shows previously published results for the Newtonian case ($\nu = 1$). As seen, our results agree with the available most accurate calculations of Ref. [13].

For the non-Newtonian case, but still without viscous dissipation, we show a comparison for $\nu = 0.5$ in Table 2.

4.2. Constant wall flux (different from zero)

Similarly well established are the results for the constant wall flux case. Then the reference temperature is selected according to

$$\Delta T_1 = \frac{q_w R_w}{\lambda} \tag{31}$$

where q_w is the prescribed (non-zero) heat flux. Now the Neumann boundary condition is recovered from (18) by putting

$$c_1 = 0; \quad c_2 = 1; \quad c_3 = 1 \tag{32}$$

The wall temperature is obtained from the solution

$$t_w = t(x, 1) \tag{33}$$

If $Br = 0$, then the bulk temperature is related to the location according to $t_b = 2x$, and hence

$$Nu_{H,x} = \frac{2}{t_w - 2x} \tag{34}$$

We can define a mean Nusselt number again as the integral average of $Nu_{H,x}$, therefore

$$Nu_{H,m} = \frac{2}{\bar{t}_w - x} \tag{35}$$

where

$$\bar{t}_w = \frac{1}{x}\int_0^1 t_w \, dx \tag{36}$$

The integral can be calculated directly, inverting $t_{s,w}/s$. This is one of the advantages of the Laplace transform approach.

Table 3 shows comparison for the Newtonian case and Table 4 for $\nu = 0.5$, both with the results of Ref. [13].

4.3. Viscous dissipation in adiabatic regime

Consider the special case of no heat flux. In other words, the viscous dissipation is heating up the fluid. For the first sight, the selection of ΔT_1 is not trivial. However, integrating the heat source term we can show that in the adiabatic regime the bulk temperature rises linearly:

Table 2
Constant wall temperature, $Br = 0$, $\nu = 0.5$

x	t_b [13]	t_b Present	$Nu_{T,x}$ Present	$Nu_{T,m}$ Present
0.001	–	0.040989	13.7422	20.9266
0.002	–	0.063861	10.8468	16.4979
0.005	0.11373	0.113731	7.99172	12.0735
0.010	0.17429	0.174291	6.41831	9.57567
0.020	0.26385	0.263854	5.25955	7.65818
0.050	0.44318	0.443181	4.29693	5.85514
0.100	0.63051	0.630508	3.99836	4.97813
0.200	–	0.832700	3.95041	4.46992
0.500	–	0.984356	3.94942	4.15767
1.0	–	0.999699	3.94942	4.05354

Table 3
Constant wall flux, $Br = 0$, $\nu = 1$

x	t_w [13]	t_w Present	$Nu_{H,x}$ Present	$Nu_{H,m}$ Present
0.001	–	0.130480	15.5666	20.9415
0.002	0.16751	0.167513	12.2314	16.4747
0.005	0.23517	0.235169	8.88222	11.9786
0.010	0.30689	0.306892	6.97127	9.40513
0.020	0.40530	0.405301	5.47493	7.38420
0.050	0.60225	0.602249	3.98209	5.36733
0.100	0.84308	0.843077	3.11005	4.21177
0.200	1.25716	1.25717	2.33327	3.25809
0.500	2.45833	2.45833	1.37143	2.12664
1.00	–	4.45833	0.81356	1.37989

Table 4
Constant wall flux, $Br = 0$, $\nu = 0.5$

x	t_w [13]	t_w Present	$Nu_{H,x}$ Present	$Nu_{H,m}$ Present
0.001	–	0.121817	16.6921	22.4741
0.002	–	0.156639	13.1028	17.6656
0.005	0.22053	0.220532	9.49973	12.8271
0.010	0.28865	0.288648	7.44467	10.0587
0.020	0.38273	0.382728	5.83553	7.88513
0.050	0.57302	0.573023	4.22812	5.71531
0.100	0.80902	0.809023	3.28395	4.46918
0.200	–	1.22059	2.43728	3.43453
0.500	–	2.42143	1.40704	2.20818
1.0	–	4.42143	0.82596	1.41485

$$\frac{\Delta T_{\mathrm{b,a}}}{X} = 2\left(\frac{3\nu+1}{\nu}\right)^{\nu}\frac{\kappa R_{\mathrm{w}}^{-1-\nu}U_{\mathrm{m}}^{\nu}}{\rho c_p} \tag{37}$$

From (37) two important results follow: first,

$$t_{\mathrm{b,a}} = 4Brx \tag{38}$$

whatever reference temperature difference is selected in (11).

Second, if we select the reference temperature difference to coincide with $T_{\mathrm{b}} - T_0$ at location $X = PeR_{\mathrm{w}}$ (that is at $x = 1$) then

$$\Delta T_1 = \frac{2\kappa}{\rho c_p}\left(\frac{3\nu+1}{\nu}\right)^{\nu} R_{\mathrm{w}}^{-\nu}U_{\mathrm{m}}^{\nu} \tag{39}$$

and the Br number becomes unity.

As a consequence, we obtain

$$t_{\mathrm{b,a}} = 4x \tag{40}$$

Note that with our scaling we avoid the problem of infinite Brinkman numbers and the Nusselt number is automatically zero in the adiabatic regime.

A next step in testing our algorithm is therefore to check the behavior in the adiabatic regime. In (18) now we set

$$c_1 = 0; \quad c_2 = 1; \quad c_3 = 0 \tag{41}$$

and with various Brinkman numbers we solve for t_{b}. The results (not shown here) confirm that with a moderate number of terms ($n = 20$) we can reproduce $t_{\mathrm{b}} = 4x$ with more than six decimal digits accuracy for a Newtonian fluid. For a non-Newtonian fluid, the number of required terms in the Galerkin method did increase for very low flow behavior indices (for $\nu = 0.001$ we needed $n = 40$ terms to achieve the same accuracy).

Fig. 1 shows the development of the adiabatic temperature profile for the Newtonian case and Fig. 2 for a highly non-Newtonian case ($\nu = 0.1$), both for $Br = 1$. As seen, the adiabatic profiles are very similar, almost independent of the rheology.

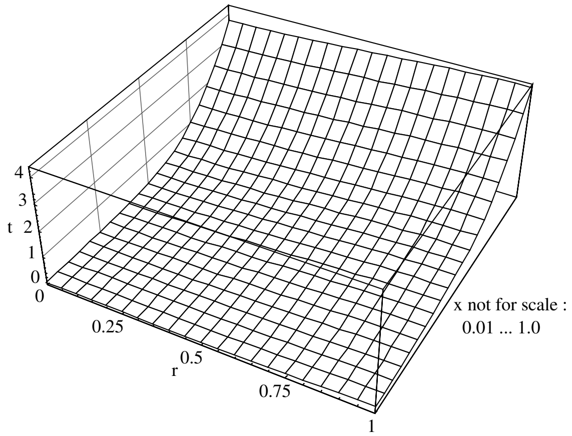

Fig. 1. Evolution of the temperature profile in adiabatic regime, Newtonian fluid ($\nu = 1$).

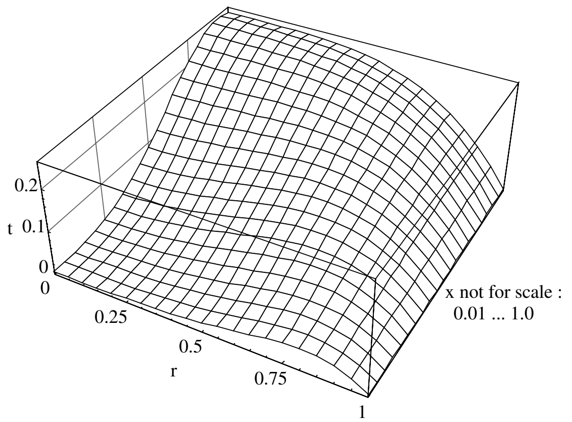

Fig. 2. Evolution of the temperature profile in adiabatic regime, flow behavior index, $\nu = 0.1$.

5. Viscous dissipation with natural cooling

Making use of the natural scale of the temperature provided by (39) we are able to study the situation called natural cooling (the Graetz–Brinkman problem). In this regime the wall temperature is kept at the ambient value. We set the constants in (18) as

$$c_1 = 1; \quad c_2 = 0; \quad c_3 = 0 \tag{42}$$

From the previous considerations it is clear, that the heat flux can be obtained from the derivative of t_{b} alone. In addition, the driving temperature difference appearing in the Nusselt number is t_{b} itself. Consequently, whatever ΔT_1 is selected in (11), the local Nusselt number can be calculated from

$$Nu_{0,x} = \frac{2Br - \frac{1}{2}\frac{\mathrm{d}t_{\mathrm{b}}}{\mathrm{d}x}}{t_{\mathrm{b}}} \tag{43}$$

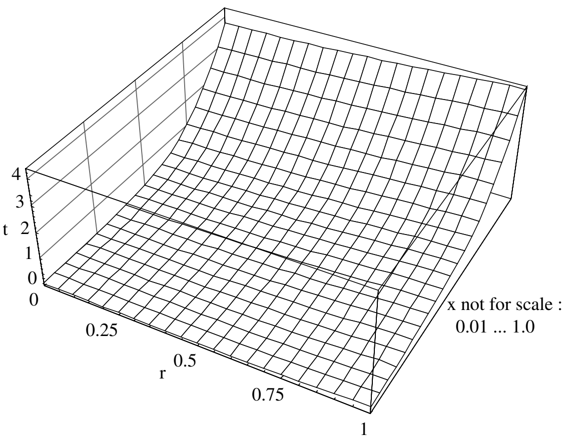

Fig. 3. Evolution of the temperature profile in natural cooling regime, Newtonian fluid ($v = 1$).

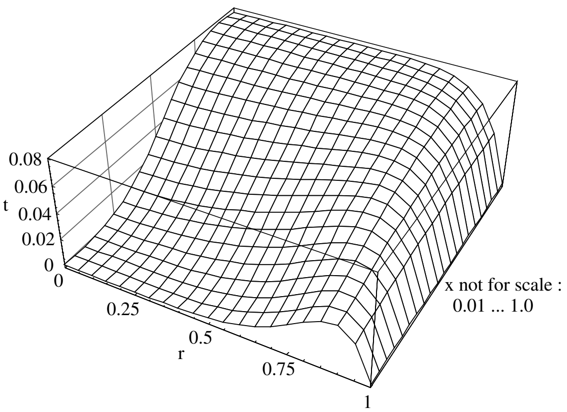

Fig. 4. Evolution of the temperature profile in natural cooling regime, flow behavior index, $v = 0.1$.

where the derivative is obtained directly, inverting $st_{s,\mathrm{b}}$. The mean Nusselt number now involves the integral average of t_b

$$Nu_{0,m} = \frac{2Br - \frac{1}{2x}t_\mathrm{b}}{\bar{t}_\mathrm{b}} \tag{44}$$

where

$$\bar{t}_\mathrm{b} = \frac{1}{x}\int_0^x t_\mathrm{b}\,\mathrm{d}x \tag{45}$$

can be again obtained directly from inverting $t_{s,\mathrm{b}}/s$.

We denote the natural-cooling Nusselt number by the subscript zero and its limiting value at infinite x by $Nu_{0,\infty}$.

Figs. 3 and 4 show the development of the temperature profiles in the natural cooling regime for the Newtonian and a highly non-Newtonian case ($v = 0.1$), respectively. In both cases we selected the reference temperature difference according to (39), and hence $Br = 1$. The basic difference between the Newtonian and the highly non-Newtonian fluids is that in the ($v = 0.1$) case most of the heat develops near to the pipe wall and the temperature profiles become steeper at the wall.

Table 5 shows the results obtained for Newtonian fluid and for a power law fluid with $v = 0.5$. As anticipated, the Nusselt number in the natural cooling regime is independent of the Brinkman number. It depends only on the location, x and flow behavior index, v. Repeating the same calculation for other flow behavior indices, we obtain a list of $(v, Nu_{0,\infty})$ pairs that can be described by the simple formula

$$Nu_{0,\infty} = \frac{2 + 16v + 30v^2}{v + 4v^2} \tag{46}$$

shown as a solid line in Fig. 5.

The above expression gives the known result: 48/5 for the Newtonian case, (see [1, p. 80]) and is in accordance with similar results obtained for the PTT fluid by Coelho et al. [18].

Table 5
Natural cooling (ambient wall temperature), any Br

x	$v = 1$			$v = 0.5$		
	t_b/Br	$Nu_{0,x}$	$Nu_{0,m}$	t_b/Br	$Nu_{0,x}$	$Nu_{0,m}$
0.001	0.003101	187.870	279.125	0.002982	220.006	327.033
0.002	0.005799	122.124	180.958	0.005522	143.297	212.482
0.005	0.012893	70.0420	103.281	0.012081	82.4309	121.691
0.010	0.022946	46.6163	68.3874	0.021187	54.9949	80.8128
0.020	0.039585	31.5111	45.8993	0.035942	37.2669	54.4052
0.050	0.076471	19.4410	27.8715	0.067764	23.0768	33.1722
0.100	0.117825	14.1374	19.7412	0.102374	16.8475	23.5743
0.200	0.164859	11.1670	14.7234	0.140142	13.3893	17.6552
0.500	0.203488	9.74152	11.4010	0.168503	11.8007	13.7654
1.0	0.208208	9.60357	10.3994	0.171372	11.6692	12.6044
10.0	0.208333	9.60000	9.67089	0.171429	11.6667	11.7507
100.	0.208333	9.60000	9.60701	0.171429	11.6667	11.6750

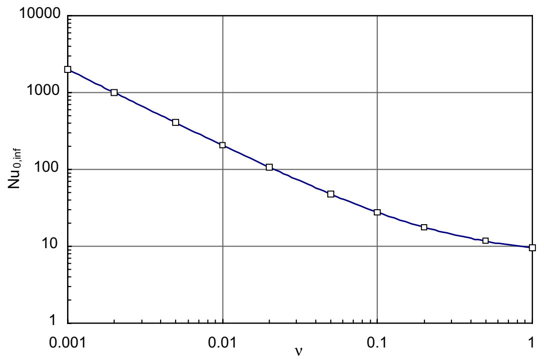

Fig. 5. Asymptotic Nusselt number in natural cooling regime as a function of flow behavior index.

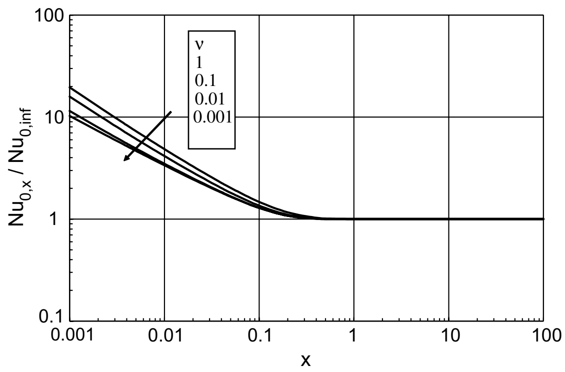

Fig. 6. Normalized local Nusselt number in natural cooling regime.

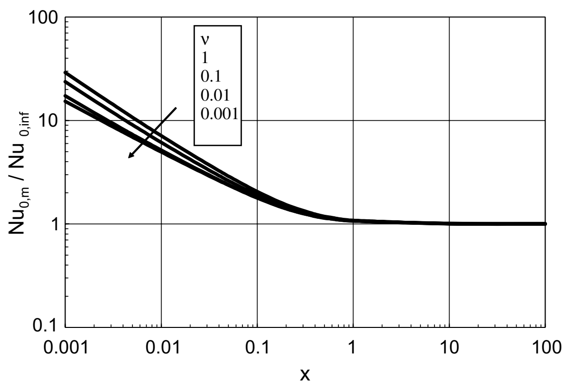

Fig. 7. Normalized mean Nusselt number in natural cooling regime.

The correlation (46) allows us to normalize the Nusselt number by its limiting value. Shown in Figs. 6 and 7 are the normalized local and mean Nusselt numbers for various flow behavior indices. The combination of the formula (46) and the figures provides both the local and the mean Nusselt numbers for any location, x and flow behavior index, ν.

6. Conclusions

The extension of the Graetz problem including viscous dissipation and power law fluid rheology model was solved using the Laplace transform Galerkin technique. The method allows for the most general boundary conditions and provides highly accurate results as shown by comparison to previously published results obtained by a variety of analytical and numerical approaches. An additional advantage of the approach is that derivatives and integrals with respect to the axial location could be handled by the operational rules of the Laplace transformation and hence no accuracy is lost in these operations.

By recognizing the internal scaling provided by the existence of the viscous energy dissipation, we introduced a new selection of the reference temperature difference. The new scaling and the flexibility of the numerical method with respect to the boundary conditions allowed us to focus on the natural cooling regime. In this regime the viscous dissipation of energy is counter-balanced by keeping the wall temperature at the ambient value. From the numerical results we derived a correlation for the asymptotic value of the Nusselt number in the natural cooling regime. The correlation reproduces the known value for the Newtonian case. By normalizing the Nusselt number with its asymptotic value, we could represent the heat transfer behavior (the local and mean Nusselt numbers) in a concise manner.

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